Close today: HW_2A, 2B (5.3,5.4)

Closes tomorrow: HW_2C (5.5)

5.5 The Substitution Rule

The Substitution Rule:

If we write u = g(x) and du = g'(x) dx, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Observations from last class

1. We are reversing the "chain rule".
2. In each case, we see
   "inside" = a function inside another
   "outside" = derivative of inside

Entry Task:

Let $u = 1 + x^4$ and correctly change the variable in

$$\int x^3(1+x^4)^{11}\,dx$$

$$u = 1 + x^4$$

$$du = 4x^3\,dx$$

$$\frac{1}{4x^3}\,du = dx$$

$$\int \cancel{x^3}\, u^{11}\, \frac{1}{4\cancel{x^3}}\,du$$

$$= \frac{1}{4}\int u^{11}\,du$$

NO MORE x's HERE!!!

$$= \frac{1}{4}\frac{1}{12}u^{12} + C$$

$$= \frac{1}{48}u^{12} + C$$

$$= \boxed{\frac{1}{48}(1+x^4)^{12} + C}$$

CHECK!!!!

$$\frac{d}{dx}:\quad \frac{1}{48}\,12\,(1+x^4)^{11}\cdot 4x^3 = x^3(1+x^4)^{11}$$

2. $\displaystyle\int \frac{\sec^2(\sqrt{x})}{\sqrt{x}}\,dx$

$u = \sqrt{x} \iff u^2 = x$

$du = \frac{1}{2\sqrt{x}}\,dx \qquad 2u\,du = dx$

$2\sqrt{x}\,du = dx$

$$\int \frac{\sec^2(u)}{\cancel{\sqrt{x}}}\, 2\cancel{\sqrt{x}}\,du \longleftarrow \boxed{\text{No more } x\text{'s!}}$$

$$2\int \sec^2(u)\,du$$

$$2\tan(u) + C$$

$$\boxed{2\tan(\sqrt{x}) + C}$$

Check!!!!

$$\frac{d}{dx} = 2\sec^2(\sqrt{x})\,\frac{1}{2\sqrt{x}} \quad \checkmark\checkmark\checkmark$$

Aside: What is really happening
(you do not need to write this)

Recall:

$$\int_a^b f(g(x))g'(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x$$

If we replace u = g(x), then we are "transforming" the problem from one involving *x* and *y* to one with *u* and *y*.

This changes ***everything*** in the set up. The lower bound, upper bound, widths, and integrand all change!

Recall from Math 124 that

$$g'(x) = \frac{du}{dx} \approx \frac{\Delta u}{\Delta x}$$

(with more accuracy when Δx is small)

Thus, we can say that

$$g'(x)\Delta x \approx \Delta u$$

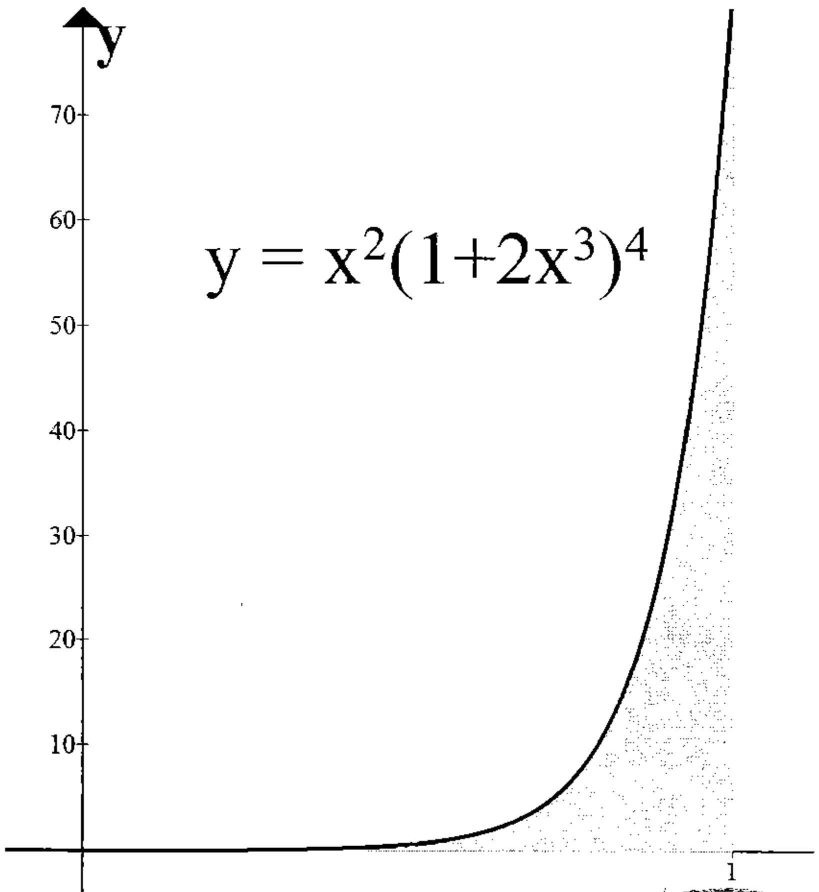
In other words, if the width of the rectangles using *x* and *y* is Δx, then the width of the rectangles using *u* and *y* is $g'(x)\Delta x$.

And if we write $u_i = g(x_i)$, then

$$\begin{aligned}\int_a^b f(g(x))g'(x)dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(g(x_i))g'(x_i)\Delta x \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} f(u_i)\Delta u \\ &= \int_{g(a)}^{g(b)} f(u)du\end{aligned}$$

Here is a visual example of this transformation

Let $u = 1 + 2x^3$.
Change *everything* in terms of *u*.



$$\int_0^1 x^2(1+2x^3)^4 dx$$

$$u = 1 + 2x^3$$

$$du = 6x^2 dx$$

$$\frac{1}{6x^2} du = dx$$

$$x = 0 \iff u = 1 + 2(0)^3 = 1$$

$$x = 1 \iff u = 1 + 2(1)^3 = 3$$

$$\int_1^3 \cancel{x^2}\, u^4 \frac{1}{6\cancel{x^2}} du$$

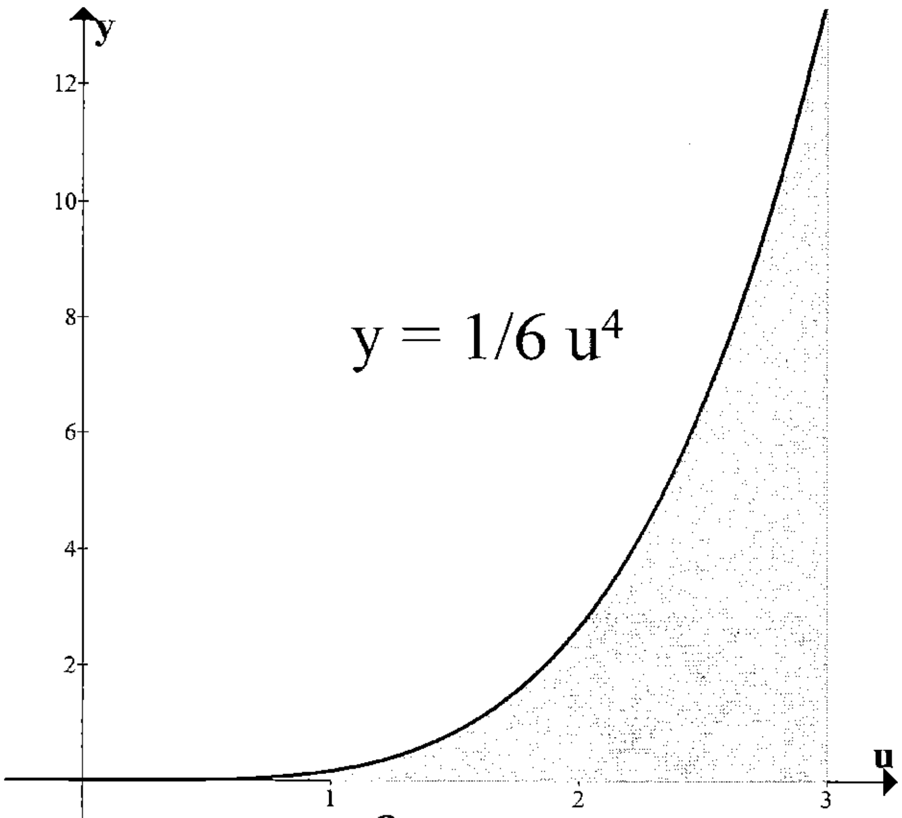
$$= \frac{1}{6} \int_1^3 u^4 du$$

ALWAYS MUST MATCH!!!

$$= \frac{1}{6} \frac{1}{5} u^5 \Big|_1^3$$

$$= \boxed{\frac{1}{30}(3^5 - 1^5)}$$

We get



$$\int_1^3 \frac{1}{6} u^4 du$$

Example: Evaluate

$$\int_2^3 x^2 e^{x^3}\, dx$$

$u = x^3$

$du = 3x^2 dx$

$\frac{1}{3x^2} du = dx$

$= \int_8^{27} \cancel{x^2} e^u \frac{1}{3\cancel{x^2}} du$

$= \frac{1}{3} \int_8^{27} e^u du$

$= \frac{1}{3} e^u \Big|_8^{27}$

$= \boxed{\frac{1}{3}\left(e^{27} - e^8\right)}$

OR (BAD WAY) SWAP BACK

$\frac{1}{3} e^{x^3} \Big|_2^3$

$\frac{1}{3} e^{(3)^3} - \frac{1}{3} e^{(2)^3}$

$\frac{1}{3} e^{27} - \frac{1}{3} e^8$

Advice on picking u = ???

Try u = *inside*

Try u = *denominator*

It doesn't take long to try something, so experiment!

$$\int x\cos(\sin(x^2))\cos(x^\wedge 2)\,dx$$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$\int \cancel{x}\cos(\sin(u))\cos(u)\frac{1}{2\cancel{x}}du$

$\frac{1}{2}\int \cos(\sin(u))\cos(u)\,du$

$t = \sin(u)$

$dt = \cos(u)\,du$

$\frac{1}{\cos(u)}dt = du$

$\frac{1}{2}\int \cos(t)\,\cancel{\cos(u)}\frac{1}{\cancel{\cos(u)}}dt$

$\frac{1}{2}\int \cos(t)\,dt = \frac{1}{2}\sin(t) + C = \boxed{\frac{1}{2}\sin(\sin(x^2)) + C}$

$$\int_0^1 \frac{x}{x^2+3}\,dx$$

$u = x^2+3$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$= \int_3^4 \frac{\cancel{x}}{u}\frac{1}{2\cancel{x}}du$

$= \frac{1}{2}\ln|u|\Big|_3^4$

$= \boxed{\frac{1}{2}\ln(4) - \frac{1}{2}\ln(3)} = \frac{1}{2}\ln(4/3)$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$\frac{1}{-\sin(x)}\,du = dx$

$$\int \frac{\cancel{\sin(x)}}{u}\,\frac{1}{-\cancel{\sin(x)}}\,du$$

$$-\int \frac{1}{u}\,du$$

$$= -\ln|u| + C$$

$$\boxed{\begin{aligned} &= -\ln|\cos(x)| + C \\ &= \ln\left|\frac{1}{\cos(x)}\right| + C \\ &= \ln|\sec(x)| + C \end{aligned}}$$

ADD TO TABLE

$$\boxed{\int \tan(x)\,dx = \ln|\sec(x)| + C}$$

What to do when the "old" variable remains:

Examples:

1. $\int x^3\sqrt{2+x^2}\,dx$

$u = 2+x^2 \Rightarrow u-2 = x^2 \Rightarrow \pm\sqrt{u-2} = x$

$du = 2x\,dx$

$\frac{1}{2x}du = dx$

$\int x^{\cancel{3}\,2}\sqrt{u}\,\frac{1}{2\cancel{x}}\,du$

WHAT NOW?!?

$\frac{1}{2}\int x^2\sqrt{u}\,du$

$\frac{1}{2}\int (u-2)\sqrt{u}\,du = \frac{1}{2}\int u^{3/2} - 2u^{1/2}\,du$

$= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2}\right) + C$

$= \boxed{\frac{1}{5}(2+x^2)^{5/2} - \frac{2}{3}(2+x^2)^{3/2} + C}$

2. $\int \frac{x^7}{x^4+1}\,dx$

$u = x^4 + 1 \Rightarrow x^4 = u - 1$

$du = 4x^3\,dx$

$\frac{1}{4x^3}du = dx$

$\int \frac{x^{\cancel{7}\,4}}{u} \frac{1}{4\cancel{x^3}}\,du$

BACK TO SUB

$\frac{1}{4}\int \frac{x^4}{u}\,du$

$\frac{1}{4}\int \frac{(u-1)}{u}\,du = \frac{1}{4}\int 1 - \frac{1}{u}\,du$

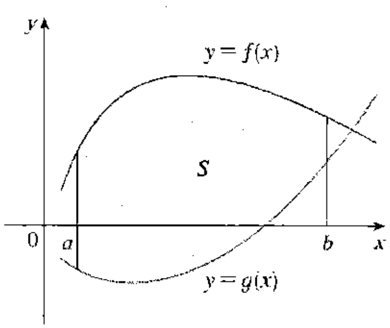
$= \frac{1}{4}(u - \ln|u|) + C$

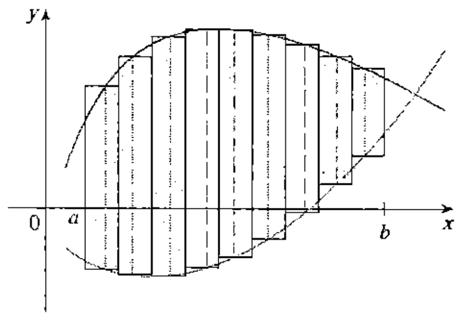
$= \boxed{\frac{1}{4}(x^4 + 1 - \ln|x^4+1|) + C}$

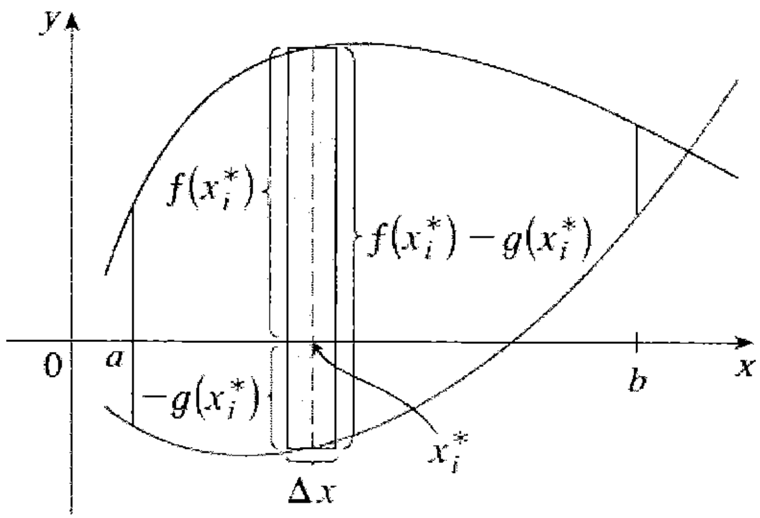
Ch 6: Basic Integral Applications

6.1 Areas Between Curves

Using dx:







(a) Typical rectangle

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} (f(x_i) - g(x_i))\Delta x$$

Example: Find the area bounded between $y = 2x$ and $y = x^2$.

$\boxed{\text{INTERSECTION}}$

$2x = x^2$

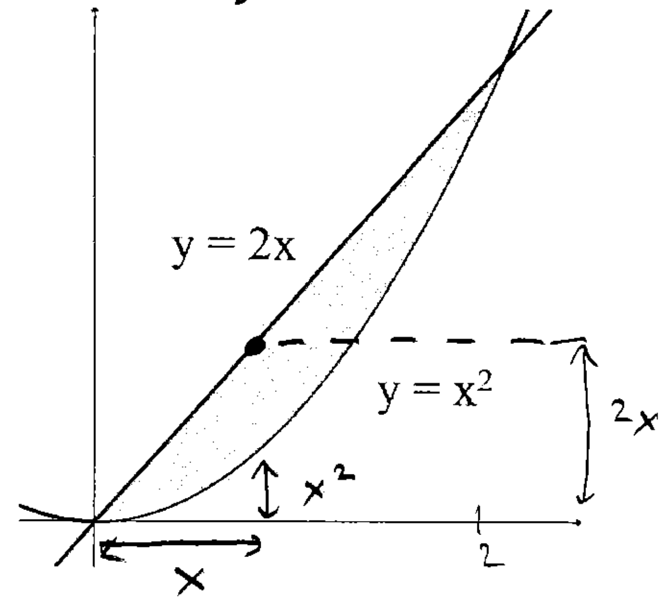
$\Rightarrow 0 = x^2 - 2x$

$0 = x(x-2)$

$x = 0$

or

$x = 2$



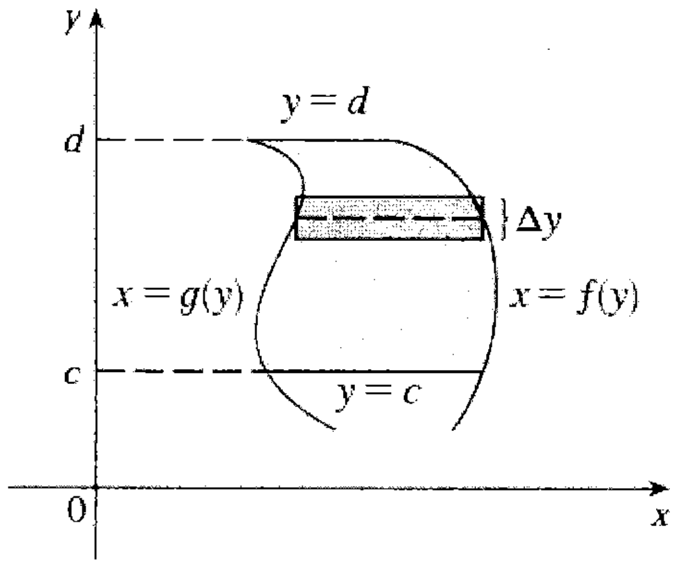
$$\int_0^2 2x - x^2 \, dx$$

$$= x^2 - \frac{1}{3}x^3 \Big|_0^2$$

$$= \left((2)^2 - \frac{1}{3}(2)^3\right) - 0$$

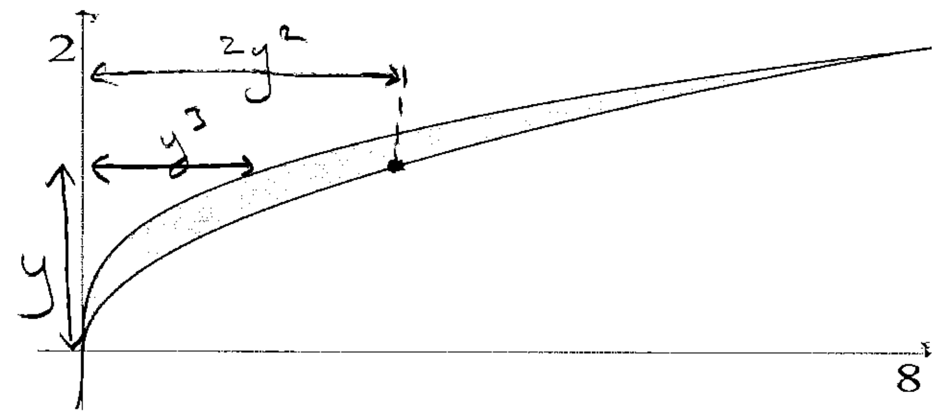
$$= 4 - \frac{8}{3} = \frac{12}{3} - \frac{8}{3} = \boxed{\frac{4}{3}}$$

Using dy:



$$\text{Area} = \lim_{n\to\infty}\sum_{i=1}^{n}\left(f(y_i) - g(y_i)\right)\Delta y$$

Example: Set up an integral for the area bounded between $x = 2y^2$ and $x = y^3$ (shown below) using dy.



$\boxed{\text{INTERSECTION}}$

$2y^2 = y^3$

$\Rightarrow 0 = y^3 - 2y^2 = y^2(y-2)$

$\Rightarrow y = 0$ or $y = 2$

$\int_0^2 2y^2 - y^3\,dy$

$\frac{2}{3}y^3 - \frac{1}{4}y^4 \Big|_0^2$

$\left(\frac{2}{3}(2)^3 - \frac{1}{4}(2)^4\right) - 0$

$\frac{16}{3} - 4 = \frac{16}{3} - \frac{12}{3} = \boxed{4/3}$

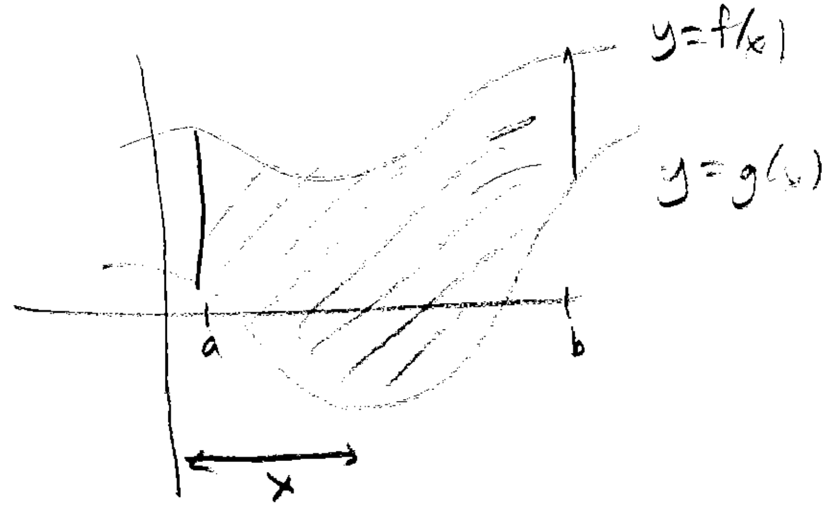
Summary: The area between curves

1. Draw picture finding all intersections.

2. Choose dx or dy. Get ***everything*** in terms of the variable you choose.

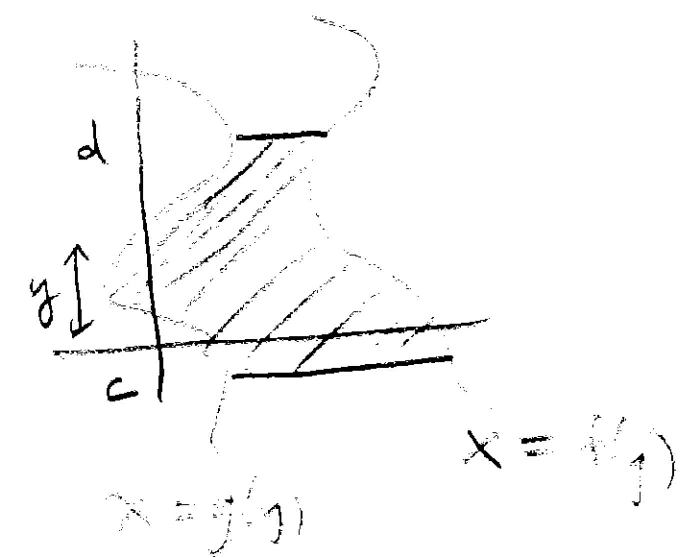
3. Draw a typical approx. rectangle.

4. Set up as follows:

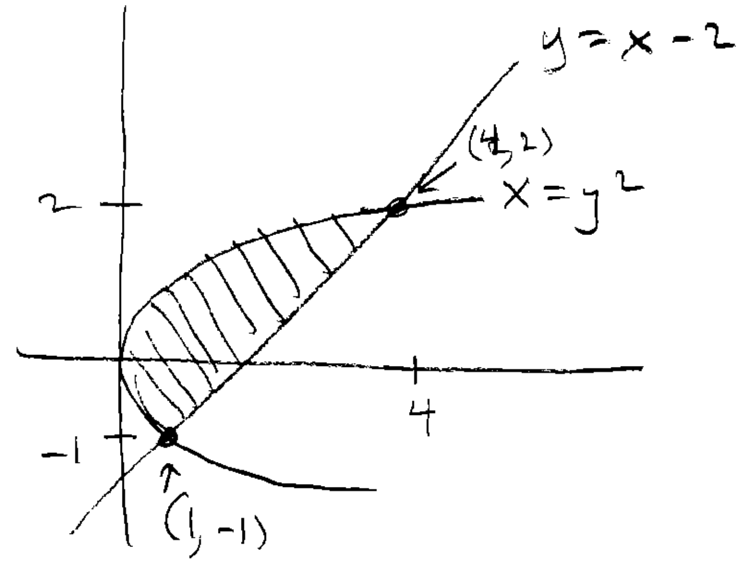


$$\text{Area} = \int_a^b (\text{TOP} - \text{BOTTOM})\, dx$$

$$\text{Area} = \int_c^d (\text{RIGHT} - \text{LEFT})\, dy$$



Example: Set up an integral (or integrals) that give the area of the region bounded by $x = y^2$ and $y = x - 2$.



INTERSECTION : $y \stackrel{?}{=} y^2 - 2$

$\Rightarrow 0 = y^2 - y - 2$

$\Rightarrow 0 = (y-2)(y+1)$

$y = -1$ or $y = 2$

EASIER TO USE dy !!!

RIGHT: $x = y + 2$

LEFT: $x = y^2$

$$\int_{-1}^{2} y + 2 - y^2 \, dy$$

$$= \frac{1}{2}y^2 + 2y - \frac{1}{3}y^3 \Big|_{-1}^{2}$$

$$= \dots = \boxed{9/2}$$

HERE IS WHAT IT LOOKS LIKE IF YOU USE dx

For $0 \le x \le 1$: TOP: $y = \sqrt{x}$ BOT: $y = -\sqrt{x}$

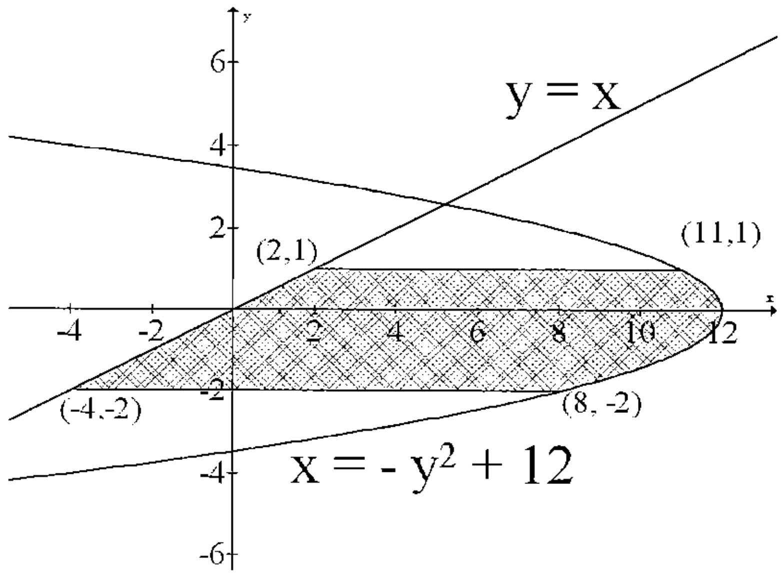
For $1 \le x \le 4$: TOP: $y = \sqrt{x}$ BOT: $y = x - 2$

$$\int_0^1 \sqrt{x} - -\sqrt{x} \, dx + \int_1^4 \sqrt{x} - (x-2) \, dx$$

$$= 2\int_0^1 \sqrt{x} \, dx + \int_1^4 \sqrt{x} - x + 2 \, dx$$

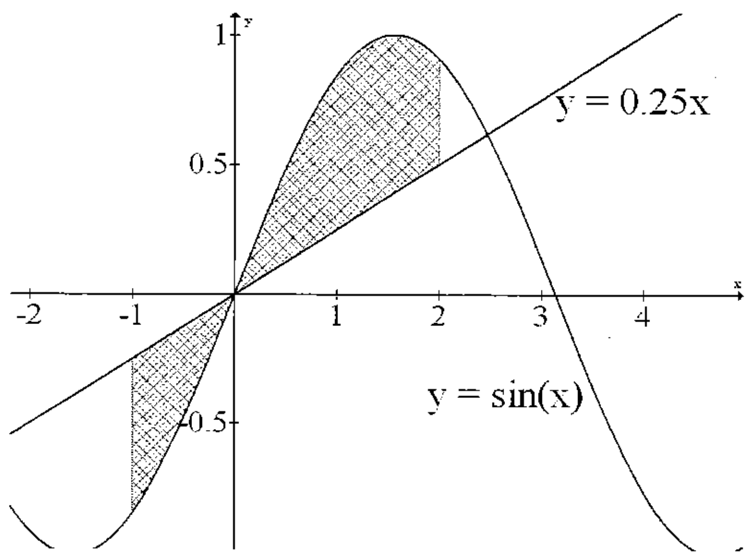
$$= \dots = \boxed{9/2}$$

Set up an integral for the total positive area of the following regions:



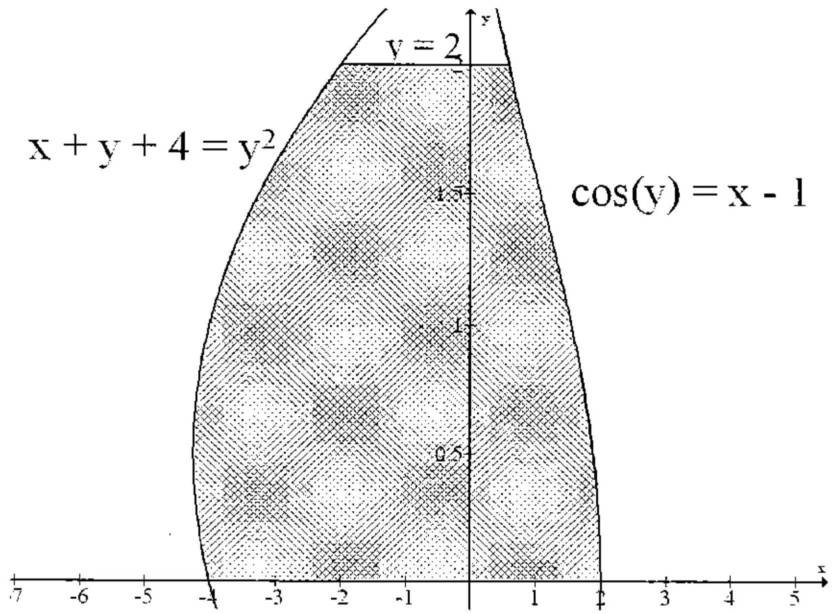
USE dy!!!

$$\boxed{\int_{-2}^{1} (-y^2+12) - y \, dy}$$



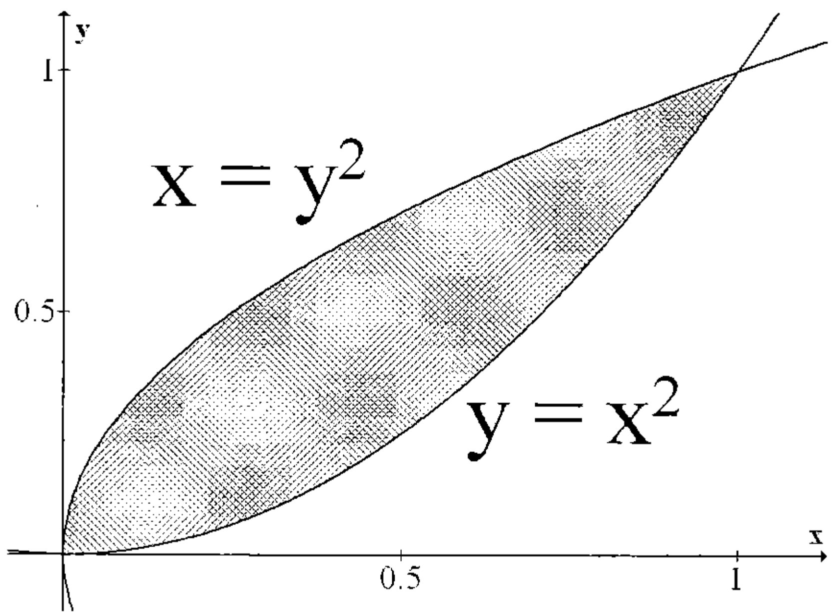
USE dx!

$$\boxed{\int_{-1}^{0} 0.25x - \sin(x)\,dx + \int_{0}^{2} \sin(x) - 0.25x \, dx}$$



USE dy!

$$\boxed{\int_0^2 (\cos(y) - 1) - (y^2 - y - 4)\, dy}$$



BOTH WORK WELL

$$\boxed{\int_0^1 \sqrt{x} - x^2\, dx}$$

OR

$$\boxed{\int_0^1 \sqrt{y} - y^2\, dy}$$